Input Uncertainty and Implications for Transient Heat Transfer in a Hollow Sphere

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Abstract: Uncertainty propagation and transient heat transfer in a hollow sphere is analyzed. The inner surface was assumed to be adiabatic and the outer surface had convective boundary condition. The stochastic Biot number, stochastic linear non-dimensional initial conditions, and various boundary conditions describe the temperature distribution throughout the hollow sphere. The resulting uncertainty amplitude was observed to have transient evolution in time. The uncertainty can either increase or decrease depending on the stochastic parameters. Results are presented for the variation of temperature due to uncertainties in the initial conditions and particular boundary conditions.

Keywords: uncertainty, transient, heat conduction, hollow-sphere.

INTRODUCTION

There are several engineering applications that need proper thermal management in the design of the system. Kennedy [1] and Patti [2] discussed the cooling of electronic systems. Bennion and Thornton [3] analyzed the cooling of electron that the properties of the systemic vehicls. Bertin and Cummings [4] discussed the greatest challenges faced with the exchange of heat on a high speed aircraft, especially with the extreme temperatures that are often associated with aerodynamic heating experienced in high speed flight.

All the previous studies assume that the properties, boundary conditions and initial conditions are fixed without accounting for any uncertainties. The modeling of aerospace systems provide a challenging problem. It is necessary to have a clear understanding and implication of the uncertainty propagation on the model output due to statistical variation of model inputs.

Forster *et al.* [5] presented uncertainty analysis to more complex, multi-variable problems. Clark *et al.* [6] developed techniques to optimize trajectories for the uncertainty of flight conditions when vehicles use PID controllers. Panasyuk and Yerkes [7] conducted a one-dimensional transient heat transfer in a thin plate with a constant heat flux boundary condition and was compared their results to a high-fidelity experimental model. This research explored the implications of thermal management for components including electronics. Panasyuk and Yerkes [8] explored a mathematical model for uncertainty analysis and commented specifically on the time evolution of the stochastic characteristics.

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Muff *et al*. [9] studied a thin plate with a constant temperature boundary condition on one surface and a convective boundary condition on the other. This is representative of a plate that may be present on a high speed vehicle. Material properties of this simulation were represented in the the Biot number, which is the ratio of internal conduction resistance to the external convective resistance of the body. With uncertainty in the material properties, the convective heat transfer coefficient, *h*, or any of the initial temperature conditions of the system, errors are expected to propagate through the model during unsteady, transient simulations. Gorla *et al.* [10,11] studied the multi-input statistical variation of a simplified linear deterministic form of a dynamic heat rejection extended surface region of a thermal management system. Dimensionless material properties of the system were represented by the Biot number and Fin number. The transient response of fins is important in a wide range of engineering devices including heat exchangers, clutches, motors, railroad roller bearings etc. The novelty in references 10 and 11 is that this is the first time the fin heat transfer was analyzed by taking into account of uncertainties in material properties of the fin, boundary conditions and initial conditions.

The present work was undertaken in order to study the multi-input statistical variation of heat rejection in a hollow spherical thermal management system. Dimensionless material properties of the system are represented by the Biot number. With uncertainty in the material properties, convective heat transfer coefficient, and initial thermal conditions of the system, errors are expected to propagate through the model during the transient heat rejection. The uncertainty in Biot number and initial conditions is critical in aerospace environments involving variable convective environments in

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high speed flight. The objective of our work is to characterize the non-dimensional temperature and heat flux to determine the factors that have the greatest risk of exceeding the known thermal management limitations. Practical application of this study can be used to simulate a fuel tank in an aircraft, thermal protection structures and high speed vehicle components.

Analysis

The governing energy equation may be written as:

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) \tag{1}$$

Next, the variables are nondimensionalized such that:

$$\theta = \frac{T(r,t)-T_{init}}{T_\infty - T_{init}}\ ,\ r^* = \frac{r-r_i}{L}, \tau = \frac{\alpha t}{L^2}\, m = \frac{r_i}{r_0} \tag{2}$$

The transformed energy equation may be written as :

$$\frac{\partial\theta}{\partial\tau} = \frac{1}{\left(\frac{m}{1-m}+r^*\right)^{*2}}\frac{\partial}{\partial r^*}\left(\left(\frac{m}{1-m}+r^*\right)^{*2}\frac{\partial\theta}{\partial r^*}\right) \tag{3}$$

A linear initial condition will be applied to the system in the nondimensional form:

$$\theta(r,0) = F_0 + F_1 r^* \tag{4}$$

Here, both F_0 and F_1 can be randomly sampled using the Monte Carlo technique. The spherical system was assumed to have adiabatic boundary at the inner radius and convective boundary condition at the outer radius.

The inner adiabatic boundary condition is described by this equation:

$$\left(\frac{d\theta}{dr^*}\right)|_{\{r^*=0\}} = 0 \tag{5}$$

and the outer convective boundary condition may be written by using Newton's Law as:

$$-k\left(\frac{dT}{dr}\right)|_{\{r=r_o\}} = h_o(T(r_o,t)-T_\infty) \tag{6}$$

then using $Bi_o = \frac{h_o r_o}{k}$, we arrive at the nondimensional outer boundary equation:

$$\left(\frac{d\theta}{dr^*}\right)|_{\{r^*=1\}} = (1-m)\, Bi_0(\theta(1,\tau)-1) \tag{7}$$

Separation of variables using $\theta(r^*,\tau) = R(r^*)\mathrm{T}(\tau)$ leads to both the time equation:

$$\frac{d\mathrm{T}}{d\tau} = -\lambda^2\mathrm{T}(\tau) \tag{8}$$

and the radial equation in dimensional form reads:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \lambda^2 R = 0 \tag{9}$$

The time equation has well known solutions of the form $\mathrm{T}(\tau) = Ae^{-\lambda^2\tau}$, where A is some constant and $-\lambda^2$ is the separation constant.

The radial differential equation can be solved using Bessel functions assuming the general form:

$$R(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r) \tag{10}$$

where C_1 and C_2 are constants and $J_0(x) = \frac{\sin(x)}{x}$ and $Y_0 = -\frac{\cos(x)}{x}$ are first order spherical Bessel functions. From here, the single mode nondimensional temperature can be written as:

$$\theta_n(r,\tau) = \left[C_{1,n}J_0(\lambda_n r) + C_{2,n}Y_0(\lambda_n r)\right]e^{-\lambda_n^2\tau} \tag{11}$$

Applying the boundary conditions described above allows us to solve for the transcendental eigenvalue roots:

$$0 = \left(\lambda J_0'(\lambda\, m) - \frac{J_0(\lambda\, m)}{m}\right)\left(\lambda Y'^{(\lambda)} + (Bi_0 \ -1)Y(\lambda)\right)$$
$$-\left(\lambda Y_0'(\lambda\, m) - \frac{Y(\lambda\, m)}{m}\right)\left(\lambda J_0'(\lambda) + (Bi_0 -1)J_0(\lambda)\right) \tag{12}$$

To satisfy both the initial and boundary conditions, $\theta_n(r^*,\tau)$ is written as a summation of all eigenfunctions:

$$\theta_n(r,\tau) = \textstyle\sum_{n=1}^{\infty} a_n R(r^*)e^{-\lambda_n^2\tau} \tag{13}$$

Finally, the values of a_n may be determined by using the initial condition $\theta(r^*,0) = F_0 + F_1 r^*$ onto the eigenfunctions by:

$$a_n = \frac{\int_0^1 \theta(r^*,0)R(r^*)(\frac{m}{1-m}+r^*)^2\, dr^*}{\int_0^1 (R(r^*))^2\left(\frac{m}{1-m}+r^*\right)^2 dr^*} \tag{14}$$

Here, a_n were numerically solved using MATLAB.

The non-dimensional temperature difference across the inner and outer faces of the hollow sphere is given by

$$\Delta\theta = \theta(0,\tau) - \theta(1,\tau) \tag{15}$$

Stochastic Simulation

After the development of the expression of the non-dimensional temperature difference across the hollow sphere, $\Delta\theta$, and the non-dimensional heat flux at the outer surface of the hollow sphere, $\theta'(1,\tau)$, Monte-Carlo Simulations were conducted to observe the behavior of the system when uncertainty was introduced to various parameters.

The three variables that had uncertainty introduced to them are the Biot number, Bi, the dimensionless constant describing initial temperature at the inner surface, F_0, and the dimensionless constant parameterizing initial temperature variation across the hollow sphere, F_1. For each term, a normal distribution was randomly generated with a standard deviation scaled appropriately to have 3 standard deviations represent +/- 10% variation in the term. In the Monte-Carlo Simulation, 100 random values were sampled for Bi, F_0, and F_1. These conditions were implemented into Equations (12) - (15) to generate the results, which were then plotted against non-dimensional time from 0 to 1000. A computer program was used to iterate calculations to find eigenvalues for each randomly selected Biot number. The stochastic variation of Biot number physically represents fluid-thermal coupling in aerospace applications involving variation in external convective heat transfer due to changes in Mach number or altitude.

RESULTS

After several simulations, data was collected and plotted against non-dimensional time. The key parameters that are varied are the Biot number, Bi_μ, and the constants $F_{0\mu}$, and $F_{1\mu}$. This provides a broad scope of material and temperature conditions that may be expected in both subsonic and high speed flow heat exchange scenarios. The results section is broken into two pieces for comparison of the temperature change across the surfaces as well as the heat flux at the outer surface.

Non-Dimensional Temperature Difference $(\Delta\theta)$ Across the Surfaces

Figures **1-3** display the non-dimensional temperature changes across the surfaces of the hollow sphere as non-dimensional time progresses. As Biot number increases, the general behavior of the non-dimensional temperature differences follows a similar trend. The transient portion of this simulation for all cases lasts until a non-dimensional time of $\tau = 1$. Beyond that point, there is a distribution of outputs, but it is much less than the uncertainty observed earlier in each simulation. Additionally, the distribution in the later portion of the simulation is approximately uniform.

As focus shifts to other aspects of the simulation, it is interesting to compare simulations as $F_{0\mu}$ and $F_{1\mu}$ change individually. Across the 45 total simulations accomplished for $\Delta\theta$, as $F_{1\mu}$ changes, the stochastic spread changes very little. This is observed by comparing the groupings of data in each individual subplot. Instead, the different $F_{1\mu}$ terms have a greater impact on the initial values of $\Delta\theta$, therefore affecting the transient path of the simulation. On the other hand, as $F_{0\mu}$ increases, the amount of uncertainty in the

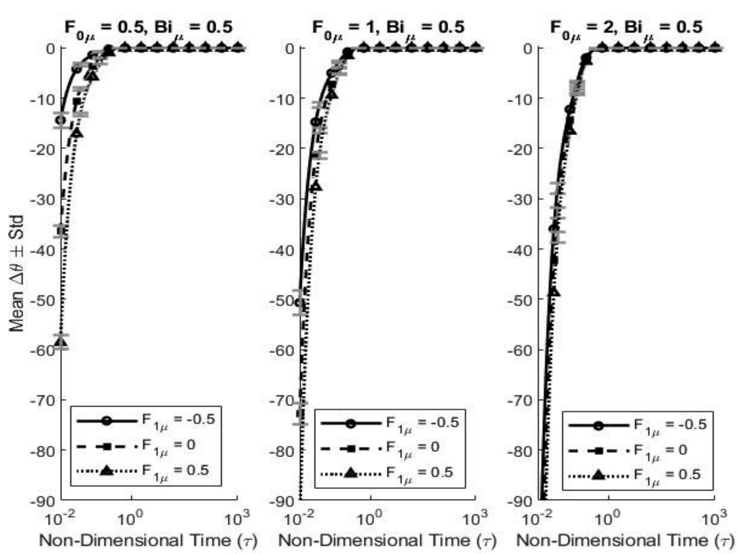

Figure 1: $\Delta\theta$ versus τ with Bi_μ = 0.5 and Variable $F_{0\mu}$ and $F_{1\mu}$.

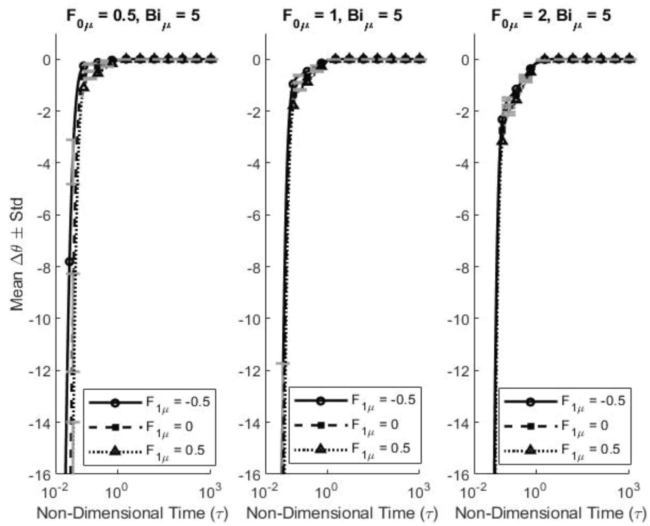

Figure 2: $\Delta\theta$ versus τ with Bi_u = 5 and Variable F_{0u} and F_{1u}

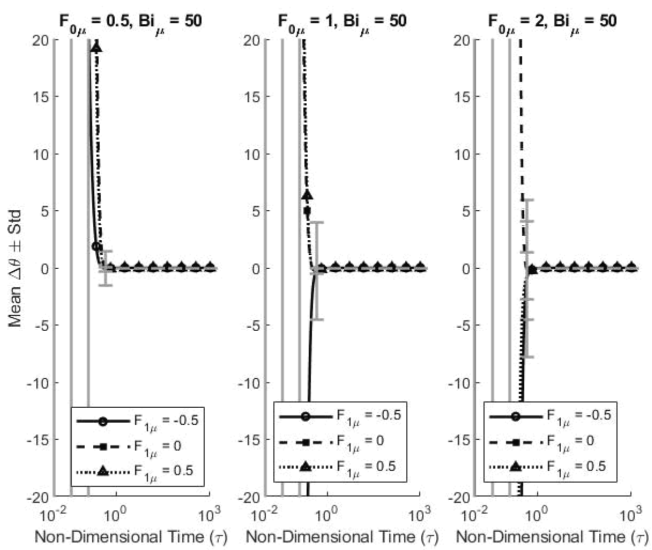

Figure 3: $\Delta\theta$ versus τ with Bi_u = 50 and Variable F_{0u} and F_{1u}

transient results also increases. This is most obvious when Biot numbers are small, such as $Bi = 0.5$ depicted in Figure **1**. In this case, the variation in $\Delta\theta$ increased from a range of about -60 to 0 for $F_{0\mu} = 0.5$ to a range of about -90 to 0 when $F_{0\mu} = 2$. This becomes less apparent as Biot number increases and the magnitude of the Biot number itself drives the non-dimensional temperature change across the cylindrical surfaces.

Non-Dimensional Heat Flux $\theta'(1,\tau)$ at the Outer Surface

The non-dimensional heat flux provides a different perspective on the one-dimensional problem than what is presented when only looking at the non-dimensional temperature difference. The parameters used for the non-dimensional heat flux are the same as those used in the non-dimensional temperature differences across the hollow sphere. Figures **4-6** are laid out and organized in the same manner as seen above.

However, in these simulations, the total uncertainty remains relatively consistent as Biot number changes. The value of $F_{1\mu}$ has little effect on the overall uncertainty of the transient response, but impacts the initial condition and path of the transients instead.

When observing the effect of $F_{0\mu}$ on $\theta'(1)$, there is a much more noticeable change. With values of $F_{0\mu} = 0.5$, the uncertainty in $\theta'(1)$ is relatively low, an the general trend is an exponential decay with all

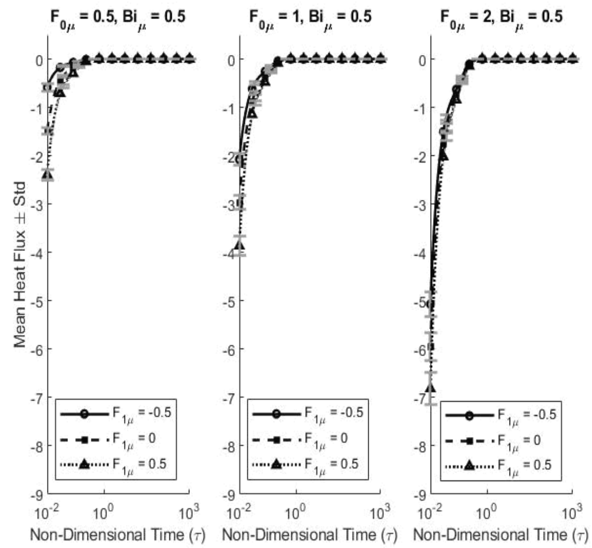

Figure 4: $\theta'(1,\tau)$ versus τ with Bi_μ = 0.5 and Variable $F_{0\mu}$ and $F_{1\mu}$.

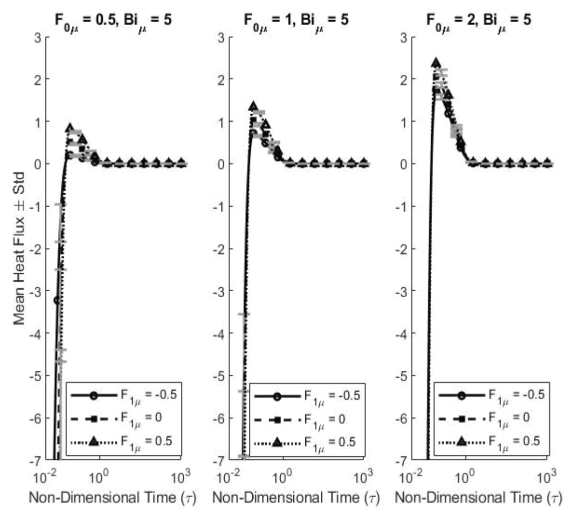

Figure 5: $\theta'(1,\tau)$ versus τ with Bi_μ = 5 and Variable $F_{0\mu}$ and $F_{1\mu}$.

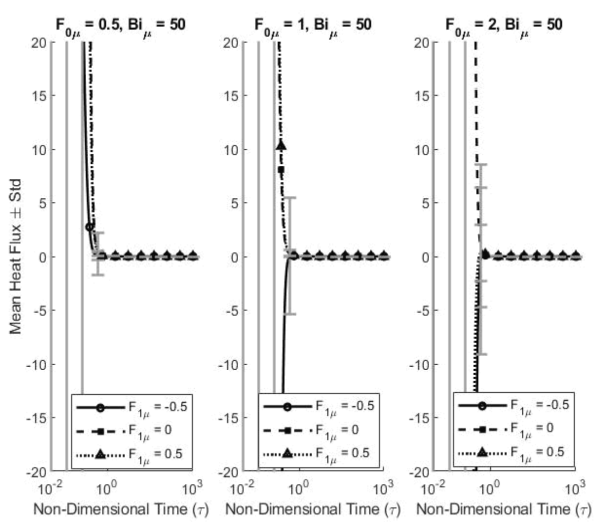

Figure 6: $\theta'(1,\tau)$ versus τ with Bi_μ = 50 and Variable $F_{0\mu}$ and $F_{1\mu}$.

negative values. With a value of $F_{0\mu} = 2$, the uncertainty in $\theta'(1)$ is also relatively higher with an exponentially decaying trend. However in this case, all values for $\theta'(1)$ are negative. This suggests that the $F_{0\mu}$ term drives the direction of heat flux in these types of simulations. Increased stochastic spread in $(\Delta\boldsymbol{\theta})$ **or** $\boldsymbol{\theta}'(\mathbf{1},\boldsymbol{\tau})$ could influence safety margins or material selection in aerospace design.

Non-Dimensional Temperature $\boldsymbol{\theta}(\mathbf{0},\boldsymbol{\tau})$ at the Inner Surface

The parameters used for the non-dimensional temperature are the same as those used in the non-dimensional temperature differences across the hollow sphere. Figures **7-9** are laid out and organized in the same manner as seen above.

However, in these simulations, the total uncertainty remains relatively consistent as Biot number changes. The value of $F_{1\mu}$ has a minor effect on the overall uncertainty of the transient response, but impacts the initial condition and path of the transients instead.

When observing the effect of $F_{0\mu}$ on $\theta(0)$, there is a much more noticeable change. With values of $F_{0\mu} = 0.5$, the uncertainty in $\theta(0)$ is relatively low, and the general trend is an exponential decay with all negative values. With a value of $F_{0\mu} = 2$, the uncertainty in $\theta(0)$ is also relatively higher with an exponentially decaying trend.

Concluding Remarks

After several Monte-Carlo simulations for the one-dimensional hollow sphere system, the behavior of the transient thermal management properties were documented and displayed. Uncertainty in the Biot number and the initial conditions of each simulation had an effect on the overall behavior of the interaction, but every simulation followed an exponentially decaying trend that ultimately reached a steady state value that had a small uniform distribution. Throughout each simulation, the uncertainty continuously decreased as time progressed.

The simulations that show the greatest variability in the values of $\Delta\theta$ are those with the highest values of $F_{0\mu}$, especially when the values of Biot numbers are low. In most instances, the uncertainty in the Biot number and the initial conditions cause outputs that can be both positive and negative, leading to an unknown for the direction of heat flux in the system. This is one of the driving factors that may impact the designs of high speed aircraft and their associated cooling systems.

This research expands upon what has been explored in several other papers for uncertainty propagation in one dimensional heat transfer scenarios with different boundary conditions. Future research can go in several different directions. First, one-dimensional problems can continue to be explored with more boundary condition configurations, or with uncertainty in more variables in the problem. Second, two-

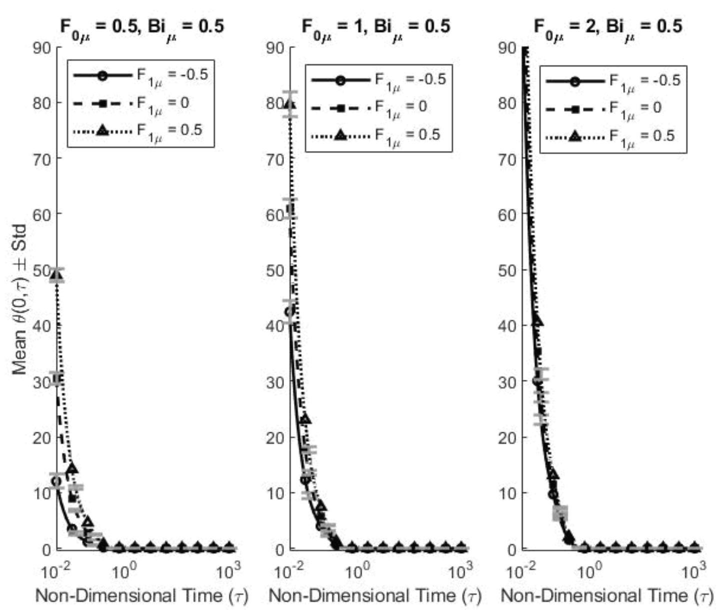

Figure 7: $\boldsymbol{\theta}(\mathbf{0},\boldsymbol{\tau})$ versus $\boldsymbol{\tau}$ with $\boldsymbol{Bi_{\mu}}$ = 0.5 and Variable $\boldsymbol{F_{0\mu}}$ and $\boldsymbol{F_{1\mu}}$.

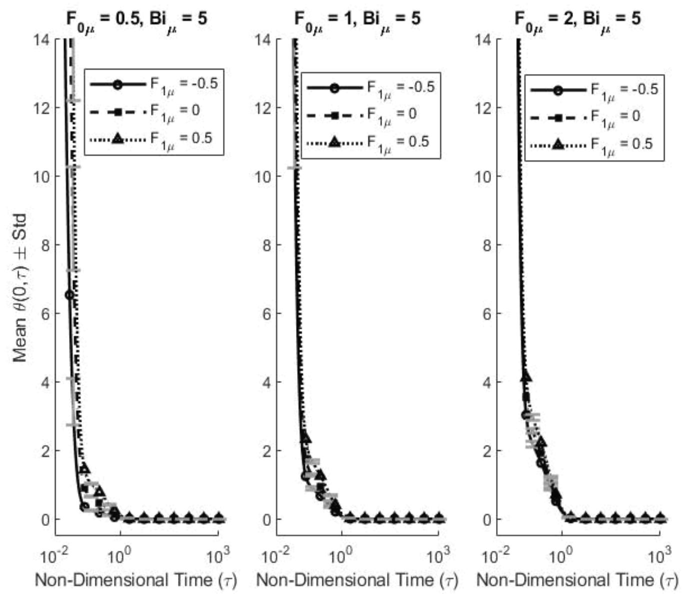

Figure 8: $\boldsymbol{\theta(0,\tau)}$ versus $\boldsymbol{\tau}$ with $\boldsymbol{Bi_{\mu}}$ = 5 and Variable $\boldsymbol{F_{0\mu}}$ and $\boldsymbol{F_{1\mu}}$.

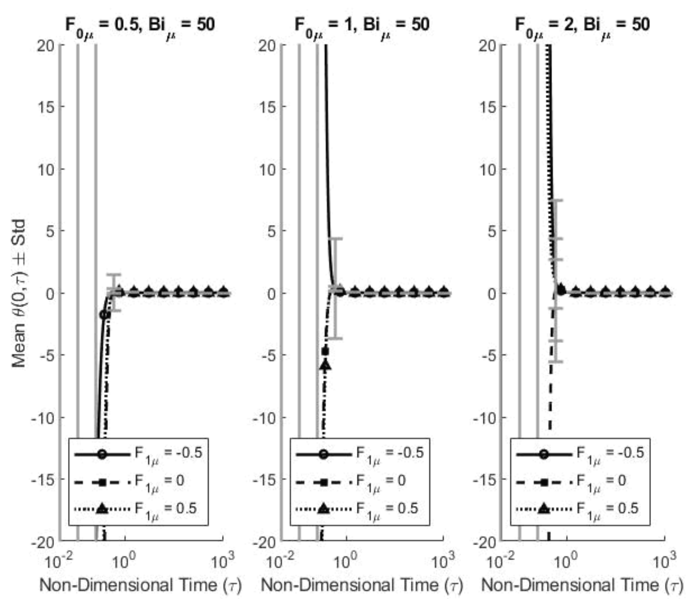

Figure 9: $\boldsymbol{\theta(0,\tau)}$ versus $\boldsymbol{\tau}$ with $\boldsymbol{Bi_{\mu}}$ = 50 and Variable $\boldsymbol{F_{0\mu}}$ and $\boldsymbol{F_{1\mu}}$.

dimensional heat transfer cases may be explored for more practical application to real life heat exchangers. Third, variable properties can be accounted for.

NOMENCLATURE

Bi = Biot number, $\frac{Lh}{k}$

Bi_{μ}= mean value for stochastic quantity Bi

F_0 = dimensionless initial temperature at inner radius, $\theta(r^* = 0, \tau = 0)$

$F_{0\mu}, F_{1\mu}$ = mean values for stochastic quantities F_0, F_1

F_1 = dimensionless constant parameterizing initial temperature variation across l

h = heat transfer coefficient, $\frac{W}{m^2K}$

J_n = n^{th} order spherical Bessel function of first kind, $J_0(x) = \frac{\sin(x)}{x}$

k = isotropic thermal conductivity of the sphere, $\frac{W}{mk}$

L = domain length, $r_o - r_i$

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Q_o= heat flux at outer radius

r = radial dimension, m

r_i, r_0 = inner and outer radii of hollow sphere respectively

r^* = radial dimensionless coordinate, $\frac{r-r_i}{L}$

T = temperature, K

T_{ref} = reference temperature, K

T_{init} = initial temperature, K

T_1 = constant parameterizing initial temperature variation across L, K

T_∞ = temperature of ambient medium, K

t = time, s

Y_n = = n^{th} order spherical Bessel function of second kind

α = thermal diffusivity, $\frac{m^2}{s}$

$\Delta\theta$ = dimensionless temperature difference

θ = dimensionless temperature

θ_{pol} = Spherical polar coordinate, radians

λ_n= eigenvalues

τ = dimensionless time, $\frac{\alpha t}{L^2}$

Subscripts

i = inner radius

$init$ = initial

n = eigenvalue index

o = outer radius

μ = mean value

REFERENCES

[1] Kennedy DP. Spreading Resistance in Cylindrical Semiconductor Devices. Journal of Applied Physics 1960; 31: 1490-1497. https://doi.org/10.1063/1.1735869

[2] Patti RS. Three Dimensional Integrated Circuits and the Future of System-on Chip Designs. Proceedings of IEEE 2006; 95: 1214-1224. https://doi.org/10.1109/JPROC.2006.873612

[3] Bennion K, Thornton M. Integrated Vehicle Thermal Management for Advanced Vehicle Propulsion Technologies. SAE Paper 2010-01-0836 2010. https://doi.org/10.4271/2010-01-0836

[4] Bertin JJ, Cummings RM. Aerodynamics for Engineers. Cambridge University Press 2022. https://doi.org/10.1017/9781009105842

[5] Forster EE, Clark DL, Beran PS. Evaluation of Designed Distributions for Stochastic Collocation Methods. AIAA SciTech Forum and Exposition 2023: 15. https://doi.org/10.2514/6.2023-0742

[6] Clark DL, Aksland CT, Lupp CA, Forster EE. Optimization under Uncertainty Techniques for Time-Dependent Aircraft Systems with Controllers 2023. https://doi.org/10.2514/6.2023-1095

[7] Panasyuk GY, Yerkes KL. Input Uncertainty and Implication for Modeling Generic and High-Fidelity Transient Convection Problems. Journal of Thermophysics and Heat Transfer 2022; 36. https://doi.org/10.2514/1.T6444

[8] Panasyuk GY, Yerkes KL. Modeling of Uncertainty Propagation for Transient Heat Rejection Problems. Journal of Thermophysics and Heat Transfer 2024; 38. https://doi.org/10.2514/1.T6820

[9] Muff JD, Gorla RSR, Forster E. Uncertainty Quantification for Transient Thermal Management. Journal of Advanced Thermal and Fluid Systems in Aerospace 2025; 1: 9-17. https://doi.org/10.66000/3110-9780.2025.01.02

[10] Gorla RSR, Forster E, Pentecost B. Uncertainty Propagation in Transient Heat Transfer from an Extended Surface. Int J Turbo & Jet Engines.

[11] Gorla RSR, Pentecost B, Forster E. Quantification of Uncertainty Propagation in Transient Heat Transfer from Flux-base Extended Surface. Int J Turbo & Jet Engines.

https://doi.org/10.66000/3110-9780.2026.02.03